III. "On Linear Differential Equations."—No. II. By W. H. L. Russell, F.R.S. Received January 20, 1870, being made up of two Papers received December 30, 1869, and January 6, 1870.

The principles laid down in my former paper will enable us to integrate a proposed differential equation, when the solution can be expressed in the form $\frac{P}{Q}\epsilon^{\omega}$, where P, Q, ω are rational and entire functions of (x).

Let

$$(\alpha_0+\alpha_1 x+\alpha_2 x^2+\ldots+\alpha_m x^m)\frac{d^n y}{dx^n}+$$

$$(\beta_0+\beta_1+\beta_2 x^2+\ldots+\beta_m x^m)\frac{d^{n-1}y}{dx^{n-1}}+$$

$$(\gamma_0+\gamma_1 x+\gamma_2 x^2+\ldots+\gamma_m x^m)\frac{d^{n-2}y}{dx^{n-2}}+\ldots$$

$$+(\lambda_0+\lambda_1+\lambda_2 x^2+\ldots+\lambda_m x^m)y=0$$

be the general linear differential equation of the nth order, where none of the indices of (x) in the coefficients of the succeeding terms are greater than those in the coefficients of the two first. Then if the equation admit of a solution of the form $\varepsilon^{\int\frac{\phi_2(x)}{\phi_1(x)}dx}$, where $\phi_1(x)$, $\phi_2(x)$ are rational and entire functions of x, and $\phi_1(x)$ and $\alpha_0+\alpha_1 x\ldots+\alpha_m x^m$ have no factors in common, and if the degree of the coefficients of the two first terms is the same,

$$y=\mathrm{E}(x)\,\epsilon^{-\int\rho dx},$$

where ρ is a root of the equation

$$\rho^n\alpha_m-\rho^{n-1}\beta_m+\rho^{n-2}\gamma_m\ldots\pm\lambda_m=0\,;$$

and if $\alpha_m=0$,

$$y=\mathrm{E}(x)\varepsilon^{-\int dx\left\{\frac{\beta_{m-1}}{\alpha_{m-1}}-\frac{\gamma_m}{\beta_m}-\frac{\alpha_{m-2}\beta_m}{\alpha^2_{m-1}}+\frac{\beta_m x}{\alpha_{m-1}}\right\}}\,;$$

and if $\alpha_m=\alpha_{m-1}=0$,

$$y=\mathrm{E}(x)$$

$$\varepsilon^{-\int dx\left\{\frac{\beta_{m-2}}{\alpha_{m-2}}-\frac{\gamma_m}{\beta_m}+\frac{\alpha_{m-3}\beta_{m-1}}{\alpha^2_{m-2}}+\frac{\alpha^2_{m-3}\beta_m}{\alpha_{m-2}}-\frac{\alpha_{m-4}\beta_m}{\alpha^2_{m-2}}+\left(\frac{\beta_{m-1}}{\alpha_{m-2}}-\frac{\alpha_{m-3}\beta_m}{\alpha^2_{m-2}}\right)x+\frac{\beta_m}{\alpha_{m-2}}x^2\right\}},$$

and so on, where the value of y is to be substituted in the proposed equation, which then becomes a linear equation to determine the rational and entire function $\mathrm{E}(x)$.

When, however, $\alpha_m=\beta_m=0$, or, in other words, when the degree of the coefficients of the succeeding terms of the proposed equation exceeds the degree of the coefficients of the two first, some modification is required;

thus if

$$(\alpha+\beta x)\frac{d^3y}{dx^3}+(\alpha'+\beta' x+\gamma' x^2)\frac{d^2y}{dx^2}+(\alpha''+\beta'' x+\gamma'' x^2+\delta'' x^3+\zeta'' x^4)\frac{dy}{dx}$$
$$+(\alpha'''+\beta''' x+\gamma''' x^2+\delta''' x^3+\zeta''' x^4+\eta''' x^5)y=0,$$
$$y=\mathrm{E}(x)\epsilon^{-\int dx\left\{\frac{\zeta'''}{\zeta''}-\frac{\delta''\eta'''}{\zeta''^2}+\frac{\gamma'\eta'''^2}{\zeta''^3}-\frac{\beta\eta'''^3}{\zeta''^4}+\frac{\eta'''}{\zeta''}x\right\}}$$

where $\mathrm{E}(x)$ is to be determined as before.

But now let $\phi_1 x$ and $\alpha_0+\alpha_1 x+\alpha_2 x^2+\ldots\alpha_m x^m$ have factors in common. We have the two equations,

$$y=\frac{\mathrm{P}}{\mathrm{Q}}\epsilon^{\omega},\quad y=\epsilon^{\int\frac{\phi_2 x}{\phi_1 x}dx},$$

$$\mathrm{PQ}\frac{dy}{dx}=\mathrm{QP}'-\mathrm{PQ}'+\mathrm{P}\omega',\quad \phi_1 x\frac{dy}{dx}=\phi_2 x;$$

hence, since $\frac{\phi_2 x}{\phi_1 x}$ is a fraction in the lowest terms, any common factors of $\phi_1 x$ and $\alpha_0+\alpha_1 x+\ldots+\alpha_m x^m$ must be factors of P or Q; hence if $x-a$ be one of the factors of $\alpha_0+\alpha_1 x+\ldots+\alpha_m x^m$, we may ascertain if it is a factor of P and Q by putting in the proposed differential equation

$$y=\mathrm{A}_m(x-a)^m+\mathrm{A}_{m+1}(x-a)^{m+1}+\mathrm{A}_{m+2}(x-a)^{m+2}+\ldots,$$

and shall thus obtain an equation to determine the index (m); and we must treat the other factors of $\alpha_0+\alpha_1 x+\alpha_2 x^2+\ldots\alpha_m x^m$ in the same way, and thus ascertain those which are also factors of P and Q.

I shall illustrate these remarks by applying them to the well-known differential equation

$$\frac{d^2u}{dx^2}-\frac{i(i+1)}{x^2}u-q^2u=0.$$

We have

$$x^2\frac{d^2u}{dx^2}-i(i+1)u-q^2x^2u=0.$$

Let

$$u=\mathrm{A}x^{\mu}+\mathrm{B}x^{\mu+1}+\mathrm{C}x^{\mu+2}+\ldots$$

Substituting $\mu(\mu-1)-i(i+1)=0$, whence $\mu=-i$; putting then $u=\frac{z}{x^i}$

$$x\frac{d^2z}{dx^2}-2i\frac{dz}{dx}-q^2xz=0;$$

hence $z=\mathrm{E}(x)\,\epsilon^{qx}$, if the equation can be integrated in the form $y=\frac{\mathrm{P}}{\mathrm{Q}}\epsilon^{\omega}$, which gives us

$$x\frac{d^2\mathrm{E}}{dx^2}+2(qx-i)\frac{d\mathrm{E}}{dx}-2iq\mathrm{E}=0.$$

Putting

$$\mathrm{E}(x)=a_0+a_1x+a_2x^2+\ldots,$$

we have

$$m(m-2i-1)a_m+2q(m-i-1)a_{m-1}=0,$$

which determines the function $E(x)$, a rational and entire function of the ith degree.

I conclude this paper with a proposition of much importance in the theory of linear differential equations,

Let

$$\phi_n x \frac{d^n y}{dx^n} + \phi_{n-1} x \frac{d^{n-1} y}{dx^{n-1}} + \phi_{n-2} x \frac{d^{n-2} y}{dx^{n-2}} + \ldots + \phi_0 x y = 0$$

be any linear differential equation. Then in general this equation will not admit a solution of the form $y = f(\epsilon^x)$. For then, putting for (x) successively $x + 2\pi i$, $x + 4\pi i$, ..., we should have

$$\phi_n(x) \frac{d^n f(\epsilon^x)}{dx^n} + \phi_{n-1} x \frac{d^{n-1} f(\epsilon^x)}{dx^{n-1}} + \phi_{n-2} x \frac{d^{n-2} f(\epsilon^x)}{dx^{n-2}} + \ldots = 0,$$

$$\phi_n(x + 2\pi i) \frac{d^n f(\epsilon^x)}{dx^n} + \phi_{n-1}(x + 2\pi i) \frac{d^{n-1} f(\epsilon_x)}{dx^{n-1}} + \ldots = 0,$$

$$\phi_n(x + 4\pi i) \frac{d^n f(\epsilon^x)}{dx^n} + \phi_{n-1}(x + 4\pi i) \frac{d^{n-1} f(\epsilon^x)}{dx^{n-1}} + \ldots = 0.$$

And these equations can be indefinitely continued. It will be observed that this solution does not comprise integrals of the form $\frac{P}{Q} \epsilon^{\frac{u}{v}}$, where $\frac{u}{v}$ is a rational function.

February 17, 1870.

Dr. WILLIAM ALLEN MILLER, Treasurer and Vice-President, in the Chair.

The following communications were read :—

I. "On a distinct form of Transient Hemiopsia." By HUBERT AIRY, M.A., M.D. Communicated by the Astronomer Royal. Received January 6, 1870.

(Abstract.)

From a comparison of the different accounts of "Hemiopsia," "Half-vision," or "Half-blindness," given by Dr. Wollaston (Phil. Trans. 1824, p. 222), M. Arago (Annales de Chimie et de Physique, tom. xxvii. p. 102), Sir David Brewster (Phil. Mag. 1865, vol. i. p. 503, and Transactions of Royal Society of Edinburgh, vol. xxiv. part 1), the Astronomer Royal (Phil. Mag. July 1865, vol. ii. p. 19), Professor Dufour (in a letter to the Astronomer Royal), Sir John Herschel (Familiar Lectures on Scientific Subjects, p. 406, Lecture IX., and private letters), Sir Charles Wheatstone (in a private letter), Mr. Tyrrell (On the Diseases of the Eye, 1840, vol. ii. p. 231), and the author of this paper, it is plain that there are different forms of transient Hemiopsia, irrespective of the wide primary distinction